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# BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH A FRACTIONAL-ORDER DIFFERENTIAL OPERATOR

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***Abstract:*** *This article examines the theoretical foundations of boundary value problems for integrodifferential equations with fractional differential operators, as well as the existence, uniqueness, and stability of their solutions. Equations constructed using the Riemann–Liouville, Caputo, and Grünwald–Letnikov operators are analyzed, based on the concept of a fractional derivative. The increasing complexity of fractional equations with the addition of integral operators and their importance in modeling physical, biological, and technical processes are substantiated. Approaches to solving boundary value problems using functional analysis, operator theory, and numerical methods are proposed.*

***Keywords:*** *fractional derivative, integrodifferential equation, integral operator, boundary value problem, Caputo operator, Riemann–Liouville operator, stability, uniqueness, function space.*

# КРАЕВЫЕ ЗАДАЧИ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОПЕРАТОРОМ ДИФФЕРЕНЦИАЛА ДРОБНОГО ПОРЯДКА

***Аннатация:*** *В статье рассматриваются теоретические основы краевых задач для интегродифференциальных уравнений с дробными дифференциальными операторами, вопросы существования, единственности и устойчивости их решений. Анализируются уравнения, построенные с использованием операторов Римана–Лиувилля, Капуто и Грюнвальда–Летникова на основе концепции дробной производной. Обосновывается возрастание сложности дробных уравнений с добавлением интегральных операторов, их значение при моделировании физических, биологических и технических процессов. Предлагаются подходы к решению краевых задач с использованием функционального анализа, теории операторов и численных методов.*

***Ключевые слова:*** *дробная производная, интегродифференциальное уравнение, интегральный оператор, краевая задача, оператор Капуто, оператор Римана–Лиувилля, устойчивость, единственность, функциональное пространство.*

## KASR TARTIBLI DIFFERENSIAL OPERATOR QATNASHGAN INTEGRO-DIFFERENSIAL TENGLAMALAR UCHUN CHEGARAVIY MASALALAR

***Annotatsiya:*** *Ushbu maqolada kasr tartibli differensial operatorlar ishtirok etuvchi integrodifferensial tenglamalar uchun chegaraviy masalalarning nazariy asoslari, ularning yechim mavjudligi, yagonaligi va barqarorligi masalalari o'rganilgan. Maqolada fractional (kasr tartibli) hosila tushunchasiga asoslangan Riemann–Liouville, Caputo va Grunvald–Letnikov operatorlari orqali tuzilgan tenglamalar tahlil qilinadi. Tadqiqotda integral operatorlarning qo'shilishi bilan kasr tartibli tenglamalarning murakkabligi oshishi, ularning fizik, biologik va texnik jarayonlarni modellashtirishdagi ahamiyati asoslab beriladi. Shuningdek, funksional analiz, operatorlar nazariyasi va sonli metodlar yordamida chegaraviy masalalarni yechish yondashuvlari taklif etiladi.*

***Kalit so'zlar:*** *kasr tartibli hosila, integrodifferensial tenglama, integral operator, chegaraviy masala, Caputo operatori, Riemann–Liouville operatori, barqarorlik, yagonalik, funksional fazo.*

## INTRODUCTION

In recent years, fractional differential equations and integrodifferential equations have become increasingly popular in mathematical physics, mechanics, biology, economics, and engineering. This is driven by the need to more accurately model the anomalous dynamics of many natural processes, such as diffusion, heat conduction, wave processes, population growth, and currents in electrical circuits.

Since it is difficult to fully describe such processes within the framework of classical differential equations, the concept of fractional derivatives was introduced. These derivatives allow for the "memory effect" to be taken into account on temporal or spatial scales, meaning that the current state of a system also depends on its past states.

## RESEARCH METHODOLOGY

Fractional differential operators, particularly the Riemann–Liouville, Caputo, and Grünwald–Letnikov formalisms, in combination with integral operators, allow for a deeper understanding of the complex nonlinear nature of physical processes. Therefore, fractional integrodifferential equations are one of the most relevant areas of theoretical and applied mathematics today.

Boundary conditions play an important role in such equations. They define the domain, boundary values, and physical content of the solution. For example, in wave processes, boundary conditions describe the distribution of energy flows at the boundary. In the case of fractional-order operators, these conditions take on a non-traditional form, since the system's behavior depends not only on the current state but also on the integral effect over the entire time interval.

In this regard, the study of boundary value problems for integrodifferential equations with fractional differential operators is one of the fundamental problems of mathematical modeling, and research in this area is widely used in the fields of mathematical physics, mechanics and computer modeling.

In this section, we will consider the formulation of boundary value problems and their solution methods when a given integro-differential equation contains a fractional differentiation operator. In fact, such equations are a special case of equations called fractional differentiation equations.

**RESEARCH AND RESULTS**

**Issue 1.**

$$y''(x) + p_1(x)y'(x) + p_2(x)y(x) +$$
$$+p_3(x)D^{\alpha}_{ax}\omega(x)y(x) = f(x), x \in (a,b) \tag{1.1}$$

The equation is continuous in the interval and$[a,b]$

$$y(a) = k_1, y(b) = k_2 \tag{1.2}$$

Find a solution that satisfies the boundary conditions, where are the given real numbers,$\alpha, a, b, k_1, k_2$ $0 < \alpha < 1$,$a < b$; $p_1(x), p_2(x), p_3(x)$and are given functions defined on a segment, and are fractional differential operators of the form. When studying boundary value problems for equations of the form (1.1), it is necessary to use the extremum principle for fractional differential operators.$[a,b]$$p_3(x) \neq 0, \omega(x) \neq 0, x \in [a,b]; D^{\alpha}_{ax} - (3.7) D^{\alpha}_{ax}$

**Lemma 2.**If is a non-decreasing positive function satisfying the Holder condition, the inequalities are valid and is ordered, then$\omega(x), p_1(x), p_2(x), p_3(x) \in C[a,b]$$p_2(x) \leq 0, p_3(x) < 0, \forall x \in (a,b)$$\omega(x)\gamma(> \alpha)$

$$y''(x) + p_1(x)y'(x) + p_2(x)y(x) + p_3(x)D_{ax}^{\alpha}\omega(x)y(x) = 0 \qquad (1.2)$$

The solution of the integro-differential equation does not reach a positive maximum or a negative minimum in the interval ( ).$a, b$

**Proof.**Let us assume the opposite, that is, let the solution of equation (1.3) have a positive maximum (negative minimum) at the point. In this case$x_0 \in (a, b)$

$$y''(x_0) \leq 0(\geq 0), y'(x_0) = 0, y(x_0) > 0(< 0),$$
$$D_{ax}^{\alpha}\omega(x)y(x)|_{x=x_0} > 0(< 0)$$

The relationship will be appropriate. Taking this into account,

$$[y''(x) + p_1(x)y'(x) + p_2(x)y(x) + p_3(x)D_{ax}^{\alpha}\omega(x)y(x)]|_{x=x_0} < 0(> 0)$$

The inequality is true. This contradicts equality (1.3). Therefore, our hypothesis is false. The lemma is proved.

If the conditions of Lemma 3.3 are met, then the problem will have no more than one solution.$\{(1.1), (1.2)\}$

**Proof** To prove this theorem we will use equation (1.36)

$$y(a) = 0, y(b) = 0(1.3)$$

It suffices to prove that the only solution satisfying the conditions is . Assume the contrary, that is, that the problem has some solution. In this case, since , according to the Weierstrass theorem, there is a number on the interval such that the relation holds. According to (1.4) and . Then. Consequently, the function attains a positive maximum or a negative minimum at the point . According to Lemma 1.3, this cannot be. The resulting contradiction shows that our hypothesis is false. Therefore, . Theorem 1.3 is proved. Let us proceed to proving that the problem has a solution. To do this, we integrate equation (1.1) over the interval and$y(x) \equiv 0\{(1.3), (1.4)\}y_0(x) \not\equiv 0 x \in [a, b]y_0(x) \in C[a, b][a, b]x_0 \sup|y_0(x)| = |y_0(x_0)| > 0[a, b]x_0 \neq a x_0 \neq b a < x_0 < b x_0 y(x) y_0(x) \equiv 0, x \in [a, b]\{(1.1), (1.2)\}[a, x]$

$$y'(x) = \frac{d}{dx}y(x), D_{ax}^{\alpha}\omega(x)y(x) = \frac{d}{dx}D_{ax}^{\alpha-1}\omega(x)y(x) =$$
$$= \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}\omega(t)y(t)dt$$

We integrate the terms containing s by parts. Then, taking this into account and changing the order of integration in the resulting iterated integral, we arrive at the following equation:$y(a) = k_1$

$$y'(x)+p_1(x)y(x)+\int_a^x\left\{p_2(t)-p_1'(t)+\frac{\omega(t)}{\Gamma(1-\alpha)}\left[p_3(x)(x-t)^{-\alpha}-\right.\right.$$

$$\left.\left.-\int_t^x p_3'(z)(z-t)^{-\alpha}dz\right]\right\}y(t)dt=\int_a^x f(t)dt+y'(a)+k_1p_1(a)$$

where is the unknown number. We again integrate this equation over the interval:$y'(a)[a,x]$

$$y(x)+\int_a^x\{p_1(t)+[p_2(t)-p_1'(t)](x-t)+$$

$$\left.+\frac{\omega(t)}{\Gamma(1-\alpha)}\int_t^x p_3(z)(z-t)^{-\alpha}(1+x-z)dz\right\}y(t)dt=$$

$$=\int_a^x(x-t)f(t)dt+y'(a)(x-a)+k_1p_1(a)(x-a)+k_1$$

Considering that and in equality (1.5), we find the unknown in the following form:$x=by(b)=k_2y'(a)$

$$y'(a)=\left[k_2-k_1-\int_a^b\{p_1(t)+[p_2(t)-p_1'(t)](b-t)+\right.$$

$$\left.+\frac{\omega(t)}{\Gamma(1-\alpha)}\int_t^b p_3(\xi)(\xi-t)^{-\alpha}(1+b-\xi)d\xi\right\}y(t)dt+$$

$$\left.+\int_a^{\dot{o}}(b-t)f(t)dt\right](b-a)^{-1}-k_1p_1(a)$$

$y'(a)$Substituting this expression into equation (1.4), we obtain

$$y(x)+\int_a^b K_1(x,t)y(t)dt=f_1(x) \quad (1.39)$$

can be written in the form here$K_1(x,t)-\{(x,t):a\le x\le b,a\le t\le b\}$Some function, bounded and piecewise continuous on a rectangle, and some function, continuous on an interval.$f_1(x)[a,b](1.6)y(x)$— is a Fredholm integral equation of the second kind for an unknown function. If we consider a homogeneous problem,$\{(1.3),(1.4)\}$

$$y(x)+\int_a^b K_1(x,t)y(t)dt=0 \quad (1.6)$$

we have a homogeneous integral equation of the form. Since the problem has only a trivial solution, equation (1.6) also has only a trivial solution. In this case, equation (1.5) has a unique solution according to Fredholm's theorem. If$\{(1.3),(1.4)\}f(x)\in C[a,b],p_1(x)\in C^1[a,b],p_2(x),p_3(x)\in C^2[a,b]$If , then, using equation (3.39), it can be shown that . Then, since the problem and equation (1.6) are equivalent, the solution to equation (1.6) is also a solution to the problem. Problem 1.4 is completely solved.$y(x) \in C^2[a,b]\{(1.1),(1.2)\}\{(1.3),(1.4)\}$

Conclusion

The theory of integro-differential equations, which includes fractional differential operators, is currently considered an effective tool for the in-depth analysis of the dynamics of complex systems. Research shows that the addition of integral operators allows for more accurate modeling of non-standard, memory-intensive, and dispersive processes.

The use of functional analysis, operator theory, spectral methods, and numerical algorithms is essential for solving boundary value problems. Particularly when working with Caputo-type derivatives, it is necessary to define initial and boundary conditions in a way that is consistent with the physical context.

Results of the study of integrodifferential equations of fractional order:

**Existence and uniqueness of a solution**allows you to define conditions;

**Stable (stable)**creates new mathematical criteria to guarantee decisions;

**Practical modeling**(thermal conductivity, seismic waves, biomathematical processes) opens up wide possibilities for

Thus, the development of the theory of boundary value problems for fractional differential equations with integral operators is one of the main promising areas of modern mathematics, contributing to the formation of scientific methodology that is widely used in various fields of science and technology.

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