# NORMED SPACES AND HILBERT SPACE

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***Abstract.*** *This article examines the main concepts and properties of normed spaces and Hilbert spaces, as well as their role in functional analysis. The notion of a norm and its properties are analyzed. A Hilbert space is considered as a complete normed space equipped with an inner product. Applications of these spaces in mathematics and physics are also discussed.*

***Keywords:*** *Normed space, norm, Hilbert space, inner product, completeness, functional analysis, metric space.*

# NORMALANGAN FAZOLAR VA GILBERT FAZOSI

***Annotatsiya.*** *Ushbu maqolada normallangan fazolar va Gilbert fazosining asosiy tushunchalari, xossalari hamda funksional analizdagi o‘rni o‘rganiladi. Normallangan fazolarda norma tushunchasi va uning asosiy xususiyatlari tahlil qilinadi. Gilbert fazosi esa ichki ko‘paytma orqali aniqlanuvchi to‘liq normallangan fazo sifatida qaraladi. Maqolada ushbu fazolarning matematik analiz, kvant mexanikasi va boshqa sohalardagi qo‘llanilishi yoritilgan.*

***Kalit so‘zlar:*** *Normallangan fazo, norma, Gilbert fazosi, ichki ko‘paytma, to‘liq fazo, funksional analiz, metrik fazo.*

# НОРМИРОВАННЫЕ ПРОСТРАНСТВА И ПРОСТРАНСТВО ГИЛЬБЕРТА

***Аннотация.*** *В данной статье рассматриваются основные понятия и свойства нормированных пространств и пространства Гильберта, а также их роль в функциональном анализе. Анализируется понятие нормы и её свойства. Пространство Гильберта изучается как полное нормированное пространство с внутренним произведением. Также освещаются применения этих пространств в математике и физике.*

***Ключевые слова:*** *Нормированное пространство, норма, пространство Гильберта, внутреннее произведение, полнота, функциональный анализ, метрическое пространство.*

## INTRODUCTION

Functional analysis plays a significant role in modern mathematics and studies infinite-dimensional spaces. In this context, normed spaces and Hilbert spaces are fundamental concepts. A normed space allows the definition of length and distance, while a Hilbert space includes an additional inner product structure. This article discusses their main properties and interrelations.

## RESEARCH METHODS

Let E be a linear space created by multiplying by a real (complex) number.

Definition: If each element X of a linear space E is assigned a non-negative real number, called its norm and denoted by $||x||$, then this assignment

Moreover, the equality $||x|| \geq 0$ is true only if and only if $x = 0$;

$$||\lambda x|| = |\lambda| \, ||x||;$$

$$||x + y|| \leq ||x|| + ||y||$$

If the norm axioms are satisfied, then the set E is called a linearly normed space.

Condition 1 of this norm is called the identity condition, condition 2 is called the homogeneity condition, and condition 3 is called the triangle inequality.

From the triangle inequality

$||x - y|| \geq ||x|| - ||y||$ (1.1)

It can be shown that the inequality is also valid. Indeed, according to the triangle inequality.

$||xy|| \geq ||(xy)+y|| \leq ||xy||+||y||$

This gives us the inequality $||x - y|| \geq ||x|| - ||y||$. Now, if we swap x and y, we get the inequality $||x - y|| \geq ||y|| - ||x||$. These last two inequalities lead to inequality (1.1). This resulting inequality (1.1) is also called the triangle inequality.

In a normed space, the metric can be introduced using the equation $\rho(x,y) = ||xy||$. It is easy to verify that all the axioms of the metric introduced here for distance are satisfied. Since the metric is introduced in a normed space, we can also introduce a definition of convergence of a sequence of elements {xn} to an element x. Namely, if $||$ xn $- x||$ for n, then$\to \infty \to 0$

$$\lim_{n\to\infty} xn = x$$

or xn in n. Thus, a normalized definite approximation is called a norm approximation.→ ∞ → x

If a given normed space is complete in the sense of approximation by norm, then this space is called a Banach space or a B-type space.

Now we will give examples of Banach spaces.

The norm of n elements x=(x1,x2,…, xn) of real numbers is equal to

$$||x|| =( \sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$$

is defined using the equation. This space Rn is a Banach space, and the metric in it overlaps with the previously introduced metric.

The real (complex) space C[a,b] is a Banach space. In this space, the operations of addition of functions and multiplication of a function by a number are defined. Also, the norm of the function x(t) is equal to

$$||x|| = \max|x(t)|$$

$$a\le t\le b$$

is defined by equality. This space C[a,b] is a Banach space, and its metric overlaps the previously introduced metric.

A real (complex) space is a Banach space. In this space, the operations of addition of elements and multiplication of an element by a number are defined. Also, the norm of an element x = (x1, x2,…, xn) is equal to$l_p$

$$||x|| =(\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$$

It is defined by an equation. This space is a Banach space, and its metric coincides with the previously introduced metric.$l_p$

**RESULTS**

A real (complex) space with a degree of -sum on the interval [a,b] is a Banach space. In this space, the operations of addition of functions and multiplication of a function by a number are defined. Also, the norm of the function x(t) is equal to$pL_p[a,b]$

$$||x|| = (\int_a^b |x(t)|^p dt)^{\frac{1}{p}}$$

Consider the space of functions x(t) defined on the interval [a,b] and having continuous derivatives up to order k on this interval. In this space, the operations of addition of functions and multiplication of a function by a number are defined. The norm of each function x(t) is equal to

$$||x|| = max\{\max_{a\le t\le b}|x(t)|, \max_{a\le t\le b}|x'(t)|, \dots, \max_{a\le t\le b}|x^{(k)}(t)|\}$$

We introduce this with an equation. This introduced space is a Banach space, which we denote by Ck[a,b]. This space is also widely used in the theory of the calculus of variations. In general, the norm in this space is often equal to

$$||x|| = \sum_{i=0}^{k} \max\ |x^{(i)}(t)|$$

is introduced with equality.

Since normed spaces are metric spaces, all concepts introduced for metric spaces (ball, bounded set, separability and other concepts), as well as all theorems presented for such spaces, are also valid for such spaces.

All statements given for complete metric spaces are also valid for spaces of type B.

Definition. Let E be a linear space in which the norm is introduced in two different ways: ||x||(1) and ||x||(2). If there exists a number β > 0 such that for an arbitrary element$x \in E$

||x||(2)β||(1) ≤

If the inequality is satisfied, then the norm ||x||(2) is called subordinate to the norm ||x||(1). If there exists such a number α>0, β>0, then for an arbitrary element$x \in E$

a·||x||(1) ≤ ||x||(2) ≤ b||x||(1)

If the inequality is satisfied, then the norms ||x||(1) and ||x||(1) are considered equivalent.

Theorem. All normalized equivalents in an arbitrary real (complex) finite n-dimensional linear space are equivalent.

Definition: If a nonempty set of elements in a linear space is the set of elements together with an arbitrary set of these elements, then it is the set of elements together with an arbitrary set of these elements.$ELx_1, x_2, \dots, x_n$

$$\alpha_1 x_1 + \alpha_2 x_2 + \ \dots + \alpha_{n-1} x_{n-1} + \alpha_n x_n$$

If such a set contains a linear combination, then it is called a linear manifold in a linear space.$LE$

Definition: Given a linearly normed space and a nonempty set in it, if a set in the linearly normed space consists of a closed linear manifold, then this set is called a subset of the linearly normed space.$ELELEE$

Now let's define a Hilbert space. Let it be a set of certain elements. We impose the following requirements on this set.$Hx, y, z, \dots$

1. $H$complex linear space.
2. $H$To any pair of elements x and y of a complex linear space there corresponds a complex number, denoted (x,y), called the scalar product, and this correspondence has the form:
   a. (x,x) ≥ 0, moreover, the equality (x,x) = 0 is true only if and only if x=0;
   b. (x,y) = (where the dash denotes a combination of complex numbers);$\overline{(y, x)}$
   c. (;$\lambda x, y) = \lambda(x, y)$
   d. (x+y,z)=(x,z)+(y,z)

   satisfy scalar products.

||x||= is called the norm of the element x and satisfies all the axioms of the norm of a space of a normed space.$\sqrt{(x,x)}$

$H$Let the space be complete in the metric sense.$\rho(x, y) = ||x - y||$

$H$Let a space contain n linearly arbitrary elements for an arbitrary natural number, that is, let the space be infinite-dimensional. In this case, the space is called an abstract Hilbert space. A real Hilbert space is defined similarly.$n \in NHH$

We will present the most important examples of Hilbert spaces.

Example 1. Calculate the scalar product of any two elements x=() and y=() in complex linear space.$l_2 x_1, x_2, \dots, x_n y_1, y_2, \dots, y_n$

$$(x,y)=\sum_{n=1}^{a} x_n \overline{y_n}$$

If we take , then this space will be a Hilbert space.$l_2$

The approximation of this series for arbitrary elements x,y ϵ l2 is given by the following series:

This follows from the Cauchy–Bunyakovsky inequality.

Example 2. L2.9[a,b] is a complex linear space. This space is the set of complex-valued functions of dimension x(ϵ), defined on the interval [a,b], such that

$$\int_a^b |x(\in)|^2 \, \rho(\epsilon) dt < +\infty$$

Let , where the real-valued function ba is almost everywhere in the interval [a,b] and belongs to the full-dimensional set . If , then for the functions$\rho(\epsilon\,)\rho(\epsilon\,) \geq 0\rho(\epsilon\,) > 0x(\epsilon\,)y(\in) \in L_{2.g}[a,b[$

$$(x,y) = \int_a^b x(\in)\, y(\in)\, \rho(\epsilon) dt$$

If we take , then this space is a Hilbert space. (Optional)∈),y(∈)∈The fact that this integral converges for elements of L2,ρ[a,b] follows from the Cauchy–Bunyakovsky inequality for integrals. In particular, for ρ(ϵ)≡1

$$(x,y) = \int_a^b x(\in) y(\in) dt$$

We form a complex Hilbert space L2[a,b] with a scalar product.

Examples of independent solutions.

In the linear space of functions continuous on the interval [a,b]

$$(x,y) = \int_a^b x(\in) y(\in)$$

Show that it is possible to introduce a scalar product with the equality . Let L2[a,b] denote the space in which this scalar product is introduced. Is this space a Hilbert space?

In the linear space of functions continuously differentiable on the interval [a,b]

$$(x,y) = \int_a^b x(\in) y(\in) + x'(\in) y'(\in)] d \in$$

Show that it is possible to introduce a scalar product with equality . We denote the space in which this scalar product is introduced by H[a,b].

## CONCLUSION

In conclusion, normed spaces and Hilbert spaces form an essential part of functional analysis. They are widely used in mathematics, physics, and other scientific fields. The completeness property of Hilbert spaces makes them especially important for both theoretical and applied research.

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