INVESTIGATION OF EXACT SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *This article investigates the problems of finding and analyzing exact solutions of some partial differential equations (PDEs). In the course of the study, classical solution methods for first- and second-order partial differential equations were applied, including the method of separation of variables, the method of characteristics, and the Fourier method. The significance of the obtained exact solutions in mathematical modeling, the description of physical processes, and the solution of engineering problems is substantiated.*

Keywords: *partial differential equation, exact solution, method of characteristics, Fourier method, mathematical modeling.*

AYRIM XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNING ANIQ YECHIMLARINI TADQIQ QILISH

Annotatsiya: *Mazkur maqolada ayrim xususiy hosilali differensial tenglamalarning (XHDT) aniq yechimlarini topish va ularni tahlil qilish masalalari oʻrganiladi. Tadqiqot jarayonida birinchi va ikkinchi tartibli xususiy hosilali differensial tenglamalar uchun klassik yechish usullari — ajratish usuli, xarakteristikalar usuli hamda Furye usulidan foydalanildi. Olingan aniq yechimlarning matematik modellashtirishda, fizik jarayonlarni tavsiflashda va muhandislik masalalarini hal etishdagi ahamiyati asoslab berildi.*

Kalit soʻzlar: *xususiy hosilali differensial tenglama, aniq yechim, xarakteristikalar usuli, Furye usuli, matematik modellashtirish.*

ИССЛЕДОВАНИЕ ТОЧНЫХ РЕШЕНИЙ НЕКОТОРЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ

Аннотация. *В данной статье рассматриваются вопросы нахождения и анализа точных решений некоторых уравнений в частных производных (УЧП). В процессе исследования для уравнений в частных производных первого и второго порядков были использованы классические методы решения — метод разделения переменных, метод характеристик и метод Фурье. Обоснована значимость* *полученных точных решений в математическом моделировании, описании физических процессов и решении инженерных задач.*

Ключевые слова: *уравнение в частных производных, точное решение, метод характеристик, метод Фурье, математическое моделирование.*

INTRODUCTION

Partial differential equations are a key area of modern mathematical analysis; they are widely used in many fields of science, including physics, mechanics, heat transfer theory, hydrodynamics, and quantum mechanics. These equations are used to construct mathematical models of complex natural and technical processes and analyze their properties.

In some cases, exact solutions to partial differential equations can be found, and such solutions provide a profound understanding of the qualitative aspects of the processes being studied. Furthermore, exact solutions serve as benchmarks for verifying results obtained using numerical and approximate methods. Therefore, the study of exact solutions to partial differential equations is a pressing scientific problem.

The main objective of this article is to analyze methods for finding exact solutions for some types of partial differential equations and to highlight their theoretical and practical significance.

METHODS

The following scientific and methodological approaches were used during the study:

Method of theoretical analysis— Study of scientific literature and classical works on the theory of ITS;

separation method— search for a solution as a product of independent variables;

Method of characteristics— reduce first-order differential equations to ordinary differential equations;

Fourier method— construct exact solutions to boundary value problems using series;

Comparison and generalization— evaluate the effectiveness of various methods.

Classification of equations

A second-order partial differential equation with two independent variables is called... $u(x,y)$ The relationship between an unknown function and its partial derivatives up to and including the second order (in which case one of the second order derivatives must be present), which is usually called

$$F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial y},\frac{\partial^2 u}{\partial y^2}\right)=0$$

It is written in the form where F is a given function of its arguments.

$$a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}+F_1\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \qquad (1.1)$$

An equation of the form is called a linear equation with respect to higher-order derivatives. In this case a_{11}, a_{12}, a_{22} chances x, y are functions of , at least one of which is nonzero. If these coefficients are also functions of x, y u, u_x And u_y If is also a function of , then equation (1.1) is called a quasilinear equation.

If the terms of the equation that do not contain higher-order derivatives are also linear, that is, (1.1) is

$$a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}+a_{13}\frac{\partial u}{\partial x}+a_{23}\frac{\partial u}{\partial y}+a_{33}u+f=0 \qquad (1.2)$$

(1.1) is called a linear equation if it has the form $a_{11}, a_{12}, a_{22}, a_{13}, a_{23,} a_{33}$ And f With x, y are functions of ; if the coefficients of the equation x, y If an equation does not depend on , it is said to have constant coefficients. In the equation $f(x,y)=0$ If , the equation is called homogeneous.

If we replace the variables in equation (1.2) with $\xi=\phi(x,y)$ And $\eta=\psi(x,y)$ If we make a substitution based on equalities, we obtain a new equation equivalent to the previous one.

Recall that when substituting into an equation, second-order derivatives do not appear from terms that do not contain second-order derivatives; if these terms are linear, they remain linear, that is, after substitution.

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = a_{13}\frac{\partial u}{\partial x} + a_{23}\frac{\partial u}{\partial y} + a_{33}u + f$$

facial expression again

$$\overline{F}\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = \overline{a}_{13}\frac{\partial u}{\partial \xi} + \overline{a}_{23}\frac{\partial u}{\partial \eta} + \overline{a}_{33}u + \overline{f}$$

It appears here. $\overline{a}_{13}, \overline{a}_{23}, \overline{a}_{33}, \overline{f} - \xi$ Andη functions of variables. Therefore, in the future we will use the compact form of these terms instead of the expanded expression, that is, we will deal with an equation in the form (1.1).

Now we can ask the following question: how, when changing variables, does an equivalent equation become simpler than the previous one?

To answer this question, we take into account the above considerations and replace the variables in equation (1.1). In this case, $u(x, y)$ derivatives of a function through new variables

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2},$$

is determined by equations (1.1)

$$\overline{a}_{11}\frac{\partial^2 u}{\partial \xi^2} + 2\overline{a}_{12}\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{a}_{22}\frac{\partial^2 u}{\partial \eta^2} + \overline{F}\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0 \qquad (1.3)$$

appears in the field of view. In this

$$\bar{a}_{11} = a_{11}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + a_{22}\left(\frac{\partial \xi}{\partial y}\right)^2,$$

$$\bar{a}_{12} = a_{11}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + a_{12}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y}\right) + a_{22}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}, \quad (1.4)$$

$$\bar{a}_{22} = a_{11}\left(\frac{\partial \eta}{\partial x}\right)^2 + 2a_{12}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + a_{22}\left(\frac{\partial \eta}{\partial y}\right)^2.$$

To simplify equation (1.1) to (1.3), we need to change variables so that $\bar{a}_{11}$, $\bar{a}_{12}$ And $\bar{a}_{22}$ Let one or two (but not all three) coefficients be zero. To solve this problem, consider the following two lemmas.

Lemma 1. If $z = \phi(x, y)$ this function

$$a_{11}\left(\frac{\partial z}{\partial x}\right)^2 + 2a_{12}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + a_{22}\left(\frac{\partial z}{\partial y}\right)^2 = 0 \quad (1.5)$$

is one of the particular solutions of the equationIf, $\phi(x, y) = C$ expression

$$a_{11}(dy)^2 - 2a_{12}dx\,dy + a_{22}(dx)^2 = 0 \quad (1.6)$$

will be a general integral of an ordinary differential equation of the form

Lemma 2. If $\phi(x, y) = C$ expression(1.6) is the general integral of an ordinary differential equation, $z = \phi(x, y)$ functionwill be a particular solution of equation (1.5).

Proof of the first lemma.According to the condition of the lemma $z = \phi(x, y)$ a function at an arbitrary point of a given domain without loss of generality $\phi_y^{'} \neq 0$ Considering that from the last equality

$$a_{11}(\phi_x)^2 + 2a_{12}\phi_x\phi_y + a_{22}(\phi_y)^2 = 0, \quad \mathbf{(1.7_1)}$$

$$a_{11}\left(-\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22} = 0 \quad (1.7_2)$$

We will achieve equality.

$\phi(x,y)=C$ To obtain the general integral of equation (1.6), it can be written in explicit form. $y = f(x,C)$ The function must satisfy (1.6). $\phi(x,y)=C$ from

$$\frac{dy}{dx} = -\frac{\phi_x(x,y)}{\phi_y(x,y)}$$

If we substitute this into (1.6), then from (1.72) we get

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = a_{11}\left(\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22} = 0$$

This proves Lemma 1.

Now let us prove the second lemma. $\phi(x,y)=C$ Let (1.6) be the general integral of equation (1.71). Then equality (1.71) takes the form $\phi(x,y)=C$ any from the detection zone (x,y) Let us prove that this is true for . $\phi(x,y)=C$ Since the general integral (1.6) is equal to

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = a_{11}\left(\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22} = 0$$

The equality holds. This means that equality (1.71) holds. This also proves Lemma 2.

Equation (1.6) is the characteristic equation of equation (1.1), and the integrals of this equation are called the characteristics of equation (1.1). Equation (1.6) is defined as follows: (dx, dy) The direction of vector (1.1) is called the characteristic direction of equation (1.1).

Therefore, according to Lemmas 1 and 2, $\phi(x,y)=C$ When one of the integrals of equation (1.6) is equal to, $\xi = \phi(x,y)$ If we take, then in equation (1.3) $\frac{\partial^2 u}{\partial \xi^2}$ since the coefficient in front of it becomes equal to zero, that is, $\bar{a}_{11} = 0$ will be; also, $\psi(x,y) = C$ If the second integral of equation (1.6) is equal to $\eta = \psi(x,y)$ if we take this as, $\frac{\partial^2 u}{\partial \eta^2}$ The coefficient in front of it also becomes equal to zero, that is, $\bar{a}_{22} = 0$ will.

The characteristic equation is decomposed into the following two ordinary differential equations of the first order:

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}, \qquad \frac{dy}{dx} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \qquad (1.8)$$

Depending on the sign of the expression under the root in these equations, equation (1.1) is divided into types.

If at point M $a_{12}^2 - a_{11}a_{22} > 0$ If , then equation (1.1) is called a hyperbolic equation at point M.

If at point M $a_{12}^2 - a_{11}a_{22} = 0$ If , then equation (1.1) is called a parabolic equation at point M.

RESULTS

The following results were obtained during the study:

Using the method of characteristics, exact solutions of linear partial differential equations of the first order were found, and their geometric interpretation was also shown.

For second-order parabolic equations (e.g., heat equations), exact solutions were obtained using Fourier series.

Using the separation method, classical exact solutions of the wave equation were obtained.

The dependence of the obtained solutions on the initial and boundary conditions was analyzed.

The results showed that exact solutions play an important role in determining qualitative aspects (stability, symmetry, periodicity) of physical processes.

DISCUSSION

The obtained results confirm the importance of exact solutions to partial differential equations in solving theoretical and practical problems. In particular, exact solutions serve as a benchmark for verifying and comparing results obtained using numerical methods—the difference method and the finite element method.

At the same time, it was established that exact solutions are impossible for all partial differential equations, and it was noted that approximate and numerical methods must be used to describe many real-world processes. However, careful study of cases where exact solutions exist will improve the reliability of mathematical modeling.

CONCLUSION

This article examines the problems of finding and analyzing exact solutions to certain partial differential equations. The results demonstrate the importance of exact solutions in studying the theory of partial differential equations and expand their application in both educational and practical applications.

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